

# A CATEGORICAL AND DIAGRAMMATICAL APPROACH TO TEMPERLEY-LIEB ALGEBRAS

YAMAGAMI SHIGERU

Department of Mathematical Sciences  
Ibaraki University  
Mito, 310-8512, JAPAN

ABSTRACT. Algebraic basics on Temperley-Lieb algebras are proved in an elementary and straightforward way with the help of tensor categories behind them.

## 1. INTRODUCTION

Temperley-Lieb algebra is a key notion in understanding quantum symmetry of various mathematical or physical objects and a variety of investigations have been worked out since the advent of the V. Jones' celebrated work on knot invariants. There have been developed three major approaches to the subject: the Jones' original method in subfactor theory together with the associated combinatorial invariants ([11], [9], [7]), representation theory of quantum groups (see [5], [12] for example), and the geometric (diagrammatic) method due to L.H. Kauffman ([13], [3]).

In this paper, we shall present main structural analyses on Temperley-Lieb algebras, such as the criterion for semisimplicity, the description of Bratteli diagrams, the existence of  $C^*$ -structures and so on, in a quite elementary and self-contained way with the emphasis on motivational streamlines of arguments. Our main tool is a diagrammatic presentation exploited in the third approach, together with naturally associated tensor categories of planar strings (called skein category in [23] and Temperley-Lieb category in [10]), where Temperley-Lieb algebras are captured as the algebra of endomorphisms in Temperley-Lieb categories.

Widely recognized is the usefulness of such diagrammatic presentations in tensor calculus (see [23], for example) but planar strings themselves also play substantial roles in describing quantum symmetries (see [14], [23] for example).

Returning to our main subject the Temperley-Lieb category belongs to the class of rigid tensor categories and if one manipulates its monoidal structure in a quite natural way, it already turns out to provide proofs of structural results mentioned above. The method is applicable to Fuss-Catalan algebras ([3], [18]) as well (see [26] for hints on the way of arguments) but we concentrate here just on the Temperley-Lieb case in viewing its fundamental importance among other related algebraic structures.

Since the results themselves are well-known, we shall not repeat them here. Instead, we will briefly review existing approaches to the subject.

Restricted to operator algebras, all the relevant analysis was worked out in [11], where the existence of (universal) Temperley-Lieb algebras is proved with the help of operator algebras of Murray and von Neumann. In that respect, the construction is highly analytical.

On the other hand, representation theory of quantum groups has been mostly algebraic in its nature and, once one knows the relevant definitions, the whole analysis can be traced after the representation theory of ordinary compact Lie groups. One big conceptual gap here is the very definition of quantum groups, it was in fact a consequence of ceaseless efforts of many researchers.

Compared to these, the approach here is straightforward, which is a combination of graphical presentation and rigidity calculus in tensor categories: the basic idea is just to try to determine the fusion rule (Clebsch-Gordan rule) in the Temperley-Lieb categories, so it is quite elementary up to topological intuition of planar isotopy of strings.

The inductive formula for Jones-Wenzl idempotents are consequently derived as a byproduct of semisimplicity analysis. In this respect, our reasoning is reverse in its order to standard arguments in [15, Chap. 16], [23, Chap. XII].

The unitarity (positivity) criterion is also presented as a natural consequence of the present approach, which should be compared with the elaborate analysis in [8].

## 2. LINEAR CATEGORIES

By a **linear category** we shall mean a category for which hom-sets are vector spaces over a specified ground field  $\mathbb{K}$  and all the relevant operations are assumed to be  $\mathbb{K}$ -linear. Therefore, given an object  $X$  in a linear category,  $\text{End}(X) = \text{Hom}(X, X)$  is a unital  $\mathbb{K}$ -algebra with the unit given by the identity morphism  $1_X$ . A linear category is said to be finite-dimensional if all hom-sets are finite-dimensional vector spaces.

Since we have applications to physics or analysis in mind, the ground field is assumed to be the complex number field though the results can be formulated for a general field with possibly extra conditions depending on situations.

Recall that, in a linear category  $\mathcal{L}$ , a direct sum  $X_1 \oplus \cdots \oplus X_m$  is defined as an object  $X$  together with morphisms  $\alpha_j : X_j \rightarrow X$ ,  $\beta_j : X \rightarrow X_j$  satisfying  $\beta_i \alpha_j = \delta_{i,j} 1_{X_i}$  and  $1_X = \alpha_1 \beta_1 + \cdots + \alpha_m \beta_m$ . Each  $X_i$  is called a direct summand of  $X$ . Since there could be many such morphisms, we shall often write  $X \cong X_1 \oplus \cdots \oplus X_m$  if we do not worry about their specific choices.

If we take another direct sum  $Y \cong Y_1 \oplus \cdots \oplus Y_n$ , there arises the natural isomorphism of vector spaces

$$\mathrm{Hom}(X, Y) \cong \bigoplus_{i,j} \mathrm{Hom}(X_i, Y_j).$$

Given a linear category  $\mathcal{L}$ , its completion by idempotents is, by definition, a linear category  $\tilde{\mathcal{L}}$ , where objects of  $\tilde{\mathcal{L}}$  consist of a pair  $(p, X)$  with  $p \in \mathrm{End}(X)$  an idempotent and hom-sets are set to be

$$\mathrm{Hom}((p, X), (q, Y)) = q \mathrm{Hom}(X, Y) p$$

with the composition of morphisms given by the operation in  $\mathcal{L}$ .

We shall also use the notation  $pX$  to stand for the object  $(p, X)$  in  $\tilde{\mathcal{L}}$ .

The notion of semisimplicity is usually formulated as a property on abelian categories. We shall deal with a specific class of non-abelian linear categories in what follows, for which we need to talk about the semisimplicity even though. We will here introduce it in a local and algebraic manner: First we extend a linear category  $\mathcal{L}$  by adding finite sequences of objects in  $\mathcal{L}$  with the notation  $X_1 \oplus \cdots \oplus X_m$  to stand for the sequence  $\{X_1, \dots, X_m\}$  and the hom-sets among these are defined by

$$\mathrm{Hom}(X_1 \oplus \cdots \oplus X_m, Y_1 \oplus \cdots \oplus Y_n) = \begin{pmatrix} \mathrm{Hom}(X_1, Y_1) & \cdots & \mathrm{Hom}(X_m, Y_1) \\ \vdots & \ddots & \vdots \\ \mathrm{Hom}(X_1, Y_n) & \cdots & \mathrm{Hom}(X_m, Y_n) \end{pmatrix}$$

with the composition of morphisms given by matrix multiplication.

**Definition 2.1.** A finite family  $\{X_1, \dots, X_m\}$  of objects in a finite-dimensional linear category is said to be **semisimple** if the algebra

$$\mathrm{End}(X_1 \oplus \cdots \oplus X_m)$$

is semisimple.

A finite-dimensional linear category  $\mathcal{L}$  is said to be **essentially semisimple** if any finite family of objects in  $\mathcal{L}$  is semisimple.

**Lemma 2.2.** *Let  $A$  be a finite-dimensional semisimple algebra over the field  $\mathbb{C}$ .*

- (i) *Given an idempotent  $p = p^2$  in  $A$ , we can find a finite family  $\{p_i\}$  of minimal idempotents such that*

$$p = \sum_i p_i, \quad p_i p_j = \delta_{i,j} p_i,$$

*which is referred to as a resolution of  $p$  in  $A$ .*

- (ii) *For any resolution  $\{p_i\}$  of  $p$ ,  $p_i A p_j = \mathbb{C} \delta_{i,j} p_i$ .*

**Corollary 2.3.** *Given a finite-dimensional essentially semisimple  $\mathbb{C}$ -linear category  $\mathcal{L}$ , we can find a family of objects  $\{X_i\}_{i \in I}$  in its idempotent-completion  $\tilde{\mathcal{L}}$  such that  $\text{Hom}(X_i, X_j) = \mathbb{C} \delta_{i,j} X_i$  and any object in  $\tilde{\mathcal{L}}$  is isomorphic to a direct sum of finitely many objects in  $\{X_i\}$ ;*

$$X \cong \bigoplus_{i \in I} m_i X_i \quad \text{with } m_i X_i = \overbrace{X_i \oplus \cdots \oplus X_i}^{m\text{-times}}.$$

*Here the multiplicity function  $m_i$  taking values in  $\{0, 1, 2, \dots\}$  admits non-zero integers only on a finite subset of  $I$ .*

*In other words, the linear category  $\tilde{\mathcal{L}}$  is remade into a semisimple linear category by adding direct sums to  $\tilde{\mathcal{L}}$ .*

### 3. TENSOR CATEGORIES

By a tensor category, we shall here mean a finite-dimensional  $\mathbb{C}$ -linear monoidal category with the unit object  $I$  satisfying  $\text{End}(I) = \mathbb{C}1_I$ . The strictness of monoidal structure is also assumed:  $(f \otimes g) \otimes h = f \otimes (g \otimes h)$ .

Recall that, given a pair of objects  $X$  and  $Y$ ,  $X$  is a left dual of  $Y$  (or equivalently  $Y$  is a right dual of  $X$ ) if we can find a pair of morphisms  $\epsilon : X \otimes Y \rightarrow I$  and  $\delta : I \rightarrow Y \otimes X$  for which the following compositions are identities:

$$\begin{aligned} X &\xrightarrow{1 \otimes \delta} X \otimes Y \otimes X \xrightarrow{\epsilon \otimes 1} X \\ Y &\xrightarrow{\delta \otimes 1} Y \otimes X \otimes Y \xrightarrow{1 \otimes \epsilon} Y \end{aligned}$$

The object  $X$  (resp.  $Y$ ) is uniquely determined by  $Y$  (resp.  $X$ ) up to isomorphisms and denoted by  ${}^*Y$  (resp.  $X^*$ ). If we can choose  $X = Y$ , the object  $X$  is said to be **self-dual**. A tensor category  $\mathcal{T}$  is said to be **rigid** if every object  $X$  admits both left and right duals.

The operation of idempotent-completion is compatible with rigidity:

**Lemma 3.1.** *Let  $\mathcal{T}$  be a rigid tensor category. Then its idempotent-completion  $\tilde{\mathcal{T}}$  is rigid as well.*

*Proof.* Let  $Y$  be a right dual of an object  $X$  with respect to morphisms  $\epsilon : X \otimes Y \rightarrow I$  and  $\delta : I \rightarrow Y \otimes X$ . If  $p \in \text{End}(X)$  is an idempotent, the morphism  $q \in \text{End}(Y)$  defined by  $q = (1_Y \otimes \epsilon)(1_Y \otimes p \otimes 1_Y)(\delta \otimes 1_Y)$  is an idempotent and  $qY$  is a right dual of  $pX$  by the morphisms  $\epsilon(p \otimes q) = \epsilon(p \otimes 1_Y) : pX \otimes qY \rightarrow I$ ,  $(q \otimes p)\delta = (1_Y \otimes p)\delta : I \rightarrow qY \otimes pX$ .  $\square$

The next is an immediate consequence of rigidity as is well-known.

**Lemma 3.2** (Frobenius Reciprocity). *In a tensor category  $\mathcal{T}$ , if an object  $X$  admits a right dual  $X^*$ , then we have the natural isomorphism*

$$\text{Hom}(Y, Z \otimes X) \cong \text{Hom}(Y \otimes X^*, Z).$$

Let  $X$  and  $Y$  be objects in a tensor category. Then the map  $\text{End}(X) \ni a \mapsto a \otimes 1_Y \in \text{End}(X \otimes Y)$  is a unital homomorphism. To see the injectivity of this map, we observe the following:

**Lemma 3.3.** *If  $Y$  admits a right dual  $Y^*$  such that  $Y \otimes Y^* \cong I \oplus Z$  for some object  $Z$ , i.e., we can find morphism  $\delta' : I \rightarrow Y \otimes Y^*$  satisfying  $\epsilon\delta' = 1_I$ , then the algebra-homomorphism  $\text{End}(X) \rightarrow \text{End}(X \otimes Y)$  is injective.*

Recall that, given an inclusion  $A \subset B$  of finite-dimensional semisimple algebras with the common unit element, its Bratteli diagram is a bipartite graph whose vertex set is the disjoint union of the set  $\hat{A}$  of equivalence classes of simple  $A$ -modules and the set  $\hat{B}$  of equivalence classes of simple  $B$ -modules with two vertices  $i \in \hat{A}$  and  $j \in \hat{B}$  connected by  $m$ -edges, where a non-negative integer  $m$  is determined as follows: letting  ${}_A X$  and  ${}_B Y$  be simple modules representing  $i$  and  $j$ , we set

$$m = \dim \text{Hom}({}_A X, {}_A Y).$$

For the inclusion of algebras in the previous lemma, the following is, though immediate, fundamental.

**Lemma 3.4.** *Under the same assumption as in the above lemma, assume that there are objects  $\{X_i\}$  and  $\{Y_k\}$  satisfying*

$$\begin{aligned} \text{Hom}(X_i, X_j) &= \delta_{i,j} \mathbb{C} 1_{X_i}, & \text{Hom}(Y_k, Y_l) &= \delta_{k,l} \mathbb{C} 1_{Y_k}, \\ \text{Hom}(X_i, Y_k) &= \{0\} = \text{Hom}(Y_k, X_i) \end{aligned}$$

and

$$X_i \otimes Y \cong \bigoplus_j m_{ij} X_j \oplus \bigoplus_k n_{ik} Y_k.$$

Then, for an object  $X$  which is isomorphic to a direct sum of finitely many  $X_i$ 's, both of  $\text{End}(X)$  and  $\text{End}(X \otimes Y)$  are semisimple with the Bratteli diagram of the inclusion  $\text{End}(X) \subset \text{End}(X \otimes Y)$  specified by  $m_{ij}$  and  $n_{ik}$  as the numbers of edges.

#### 4. TEMPERLEY-LIEB CATEGORIES

Here we shall review the planar description of Temperley-Lieb algebras according to L.H. Kauffman ([13], [14]), which leads us to the accompanied tensor categories at the same time (cf. [23], [1], [10]).

Let  $m$  and  $n$  be non-negative integers of the same parity. Choose a rectangle in the plane with marking of  $m$  points on the upper horizontal edge and  $n$  points on the lower horizontal edge. Join these  $m+n$  points by  $(m+n)/2$  planar curves inside the rectangular box so that curves do not cross each other. We denote by  $K_{m,n}$  the set of isotopy classes of planar curves of this type. An element in  $K_{m,n}$  is referred to as a Kauffman diagram of type  $(m,n)$ . The set  $K_{n,n}$  is simply denoted by  $K_n$ . It is well-known that the set  $K_{m,n}$  consists of  $C_{(m+n)/2}$  diagrams, where  $C_n = \binom{2n}{n}/(n+1)$  denotes the  $n$ -th Catalan number. When  $m$  and  $n$  have different parity, we set  $K_{m,n} = \emptyset$ .

See Figure 1 (the bounding boxes being omitted) for the isotopy patterns in  $K_3 = K_{3,3}$ .

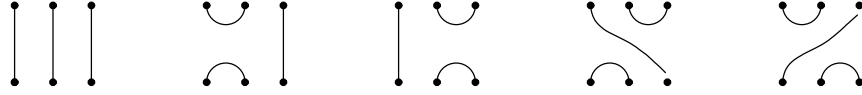


FIGURE 1.

Let  $\mathbb{C}[K_{m,n}]$  be the free  $\mathbb{C}$ -vector space generated by the set  $K_{m,n}$ , which is also denoted by  $\mathcal{K}_d(m,n)$ . Given a complex number  $d$ , a linear category  $\mathcal{K}_d$  is defined in the following way: objects of  $\mathcal{K}_d$  are non-negative integers and hom-sets are set to be  $\text{Hom}(n,m) = \mathbb{C}[K_{m,n}] = \mathcal{K}_d(m,n)$  with the composition of morphisms given by the concatenation of planar strings through one horizontal edge of boxes (diagrams stream from bottom to top by convention), where each loop (if there appeared any) is (removed and) replaced by the complex number  $d$  (Figure 2).

Taking the dependence on  $d$  into account, we also use the notation  $\mathbb{C}[K_{m,n}, d]$ .

For  $D \in K_{m,n}$  and  $D' \in K_{m',n'}$ , we define  $D \otimes D' \in K_{m+m',n+n'}$  by placing  $D$  and  $D'$  horizontally so that  $D$  is left to  $D'$  (juxtaposition).

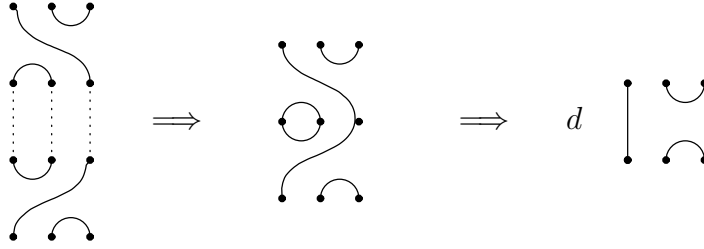


FIGURE 2.

The operation is clearly associative and is linearly extended to the map  $\mathbb{C}[K_{m,n}] \otimes \mathbb{C}[K_{m',n'}] \rightarrow \mathbb{C}[K_{m+m',n+n'}]$ , which makes  $\mathcal{K}_d$  into a tensor category, called the **Temperley-Lieb category**. The terminology is in accordance with [1], [10] though it is called skein category in [23].)

In the category  $\mathcal{K}_d$ , the multiplicative notation is also used to indicate objects; we introduce a dummy symbol  $X$  to represent the object 1 so that the object  $n$  is expressed by

$$X^{\otimes n} = \overbrace{X \otimes \cdots \otimes X}^{n\text{-times}}.$$

For short, we use the notation  $X^n$  occasionally.

The Temperley-Lieb category possesses a specific feature of perfect rigidity: Firstly, the generating object  $X$  is self-dual with respect to the pairing  $\epsilon : X \otimes X \rightarrow I$  and the copairing  $\delta : I \rightarrow X \otimes X$  given by the arcs (Figure 3).

$$\delta = \text{arc} \quad , \quad \epsilon = \text{arc} \quad .$$

FIGURE 3.

Iterating these basic morphisms, we see that every object is self-dual: The pairing and copairing of  $X^3$ , for example, are given by the diagrams in Figure 4.

$$\text{diagram} \quad , \quad \text{diagram} \quad .$$

FIGURE 4.

Frobenius transforms are then visually realized by bending terminal lines so that it changes directions of morphisms (Figure 5).

As an application of this geometrical interpretation, we see that two ways of complete bending coincide (Figure 6).

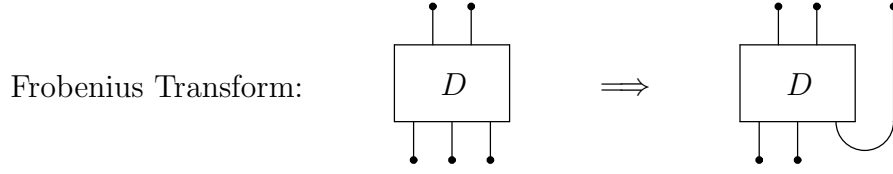


FIGURE 5.

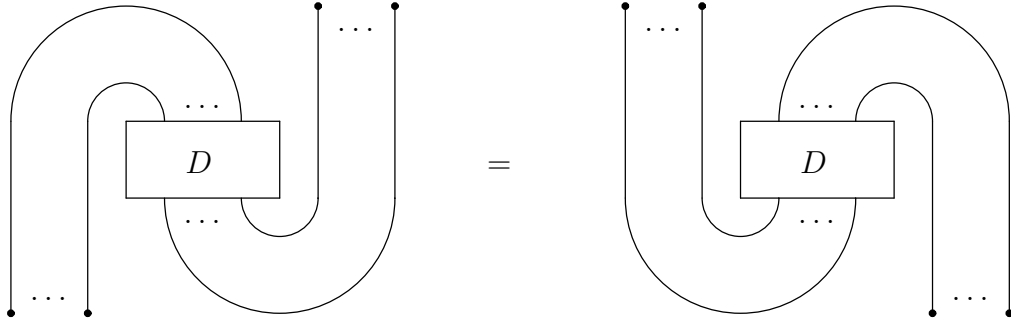


FIGURE 6.

In fact, given a diagram  $D \in K_{m,n}$ , both of these operations result in the transposed diagram  ${}^tD \in K_{n,m}$  which is, by definition, the rotation of  $D$  by an angle of  $\pi$ . See Figure 7 as an example of  $D \in K_{3,1}$  and  ${}^tD \in K_{1,3}$ .

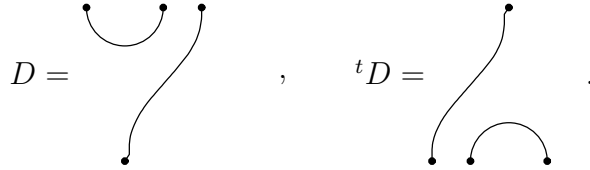


FIGURE 7.

The operation is linearly extended to the map  $\text{Hom}(X^m, X^n) \rightarrow \text{Hom}(X^n, X^m)$ , which is clearly involutive and antimultiplicative:  ${}^t({}^t f) = f$  and  ${}^t(fg) = ({}^t g)({}^t f)$ . Moreover, it is compatible with the monoidal structure in the sense that  ${}^t(f \otimes g) = {}^t g \otimes {}^t f$  for  $f \in \text{Hom}(X^m, X^{m'})$  and  $g \in \text{Hom}(X^n, X^{n'})$ , i.e., the duality holds for rigidity.

On each algebra  $\text{End}(X^n)$ , a special functional  $\text{tr}_n$  (called **Markov trace**) is defined by closing diagrams completely on  $K_n$  and then taking the linear extension to  $\text{End}(X^n) = \mathbb{C}[K_n]$ . Apparently we have two choices for closing in the plane, which, however, gives the same result as indicated by Figure 8. The suffix  $n$  is often omitted if it causes no confusion. (See [20], [6], [2] for generalities on duality and traces.)



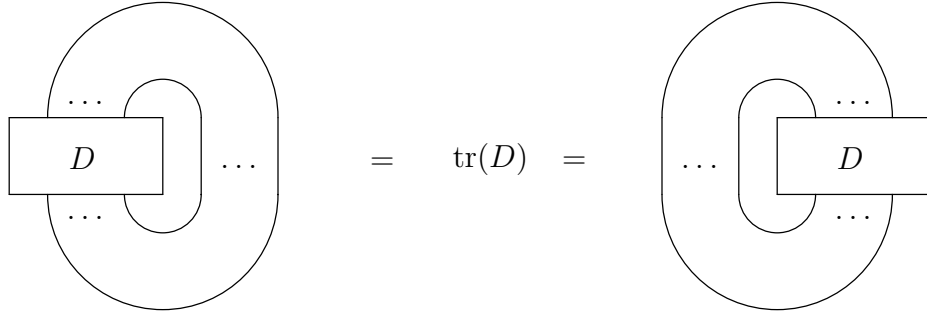


FIGURE 8.

Here are some of formulas concerning the Markov trace and transposed morphisms:

- (i)  $\text{tr}_n(fg) = \text{tr}_m(gf)$  for  $f \in \text{Hom}(X^m, X^n)$  and  $g \in \text{Hom}(X^n, X^m)$ .
- (ii)  $\text{tr}_n(h) = \text{tr}_n({}^t h)$  for  $h \in \text{End}(X^n)$ .

*Remark .* The reflection of diagrams vertically or horizontally gives another involution on hom-sets. These with the rotation as transposed morphisms constitute a symmetry of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ .

Let us now introduce **elementary diagrams**  $h_1, \dots, h_{n-1}$  in  $K_n$  by Figure 9

$$h_1 = \begin{array}{c} \text{---} \cup \text{---} \\ \cap \text{---} \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \cdots, \quad h_2 = \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \cup \text{---} \\ \cap \text{---} \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \cdots, \quad h_3 = \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \cup \text{---} \\ \cap \text{---} \end{array} \begin{array}{c} \bullet \\ \bullet \end{array} \cdots,$$

FIGURE 9.

which satisfy the relations of Temperley-Lieb in  $\text{End}(X^n) = \mathbb{C}[K_n, d]$ :

$$h_i^2 = dh_i, \quad h_i h_j = h_j h_i \quad (|i - j| \geq 2), \quad h_i h_{i\pm 1} h_i = h_i.$$

By adding one vertical line (with two end points) to the right end of planar strings in  $K_n$ , we have the imbedding  $K_n \subset K_{n+1}$ , which induces an inclusion of algebras  $\mathbb{C}[K_n, d] \subset \mathbb{C}[K_{n+1}, d]$ . In terms of the monoidal structure, this is expressed by  $a \mapsto a \otimes 1$ .

The elementary diagrams  $\{h_1, \dots, h_{n-1}\}$ , together with the unit diagram 1, turn out to generate the algebra  $\mathbb{C}[K_n, d]$  (cf. [14, Theorem 4.3]): In fact, given a diagram  $D$  in  $K_n$ , we stretch out each string in  $D$  vertically and then wave it horizontally. Then many minimal arcs are coupled by cutting strings horizontally at levels without critical crossings, resulting in a product formula for a planar diagram in terms of the elementary diagrams  $\{h_1, \dots, h_{n-1}\}$  (Figure 10). (See Appendix A for a more detailed account.)

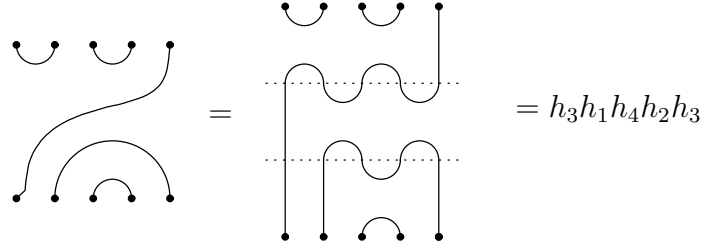


FIGURE 10.

The following can be read off from Figure 11, which already gives a pictorial proof of the formula for iterated basic constructions discussed in [21] (see [25] for the tensor-categorical meaning of the Jones basic construction).

**Example 4.1.** The  $h$ -element for the object  $X^{\otimes n}$  is given by

$$(h_n h_{n-1} \dots h_1)(h_{n+1} h_n \dots h_2) \dots (h_{2n-1} h_{2n-2} \dots h_n).$$

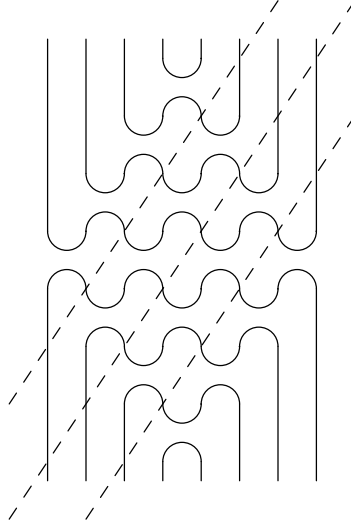


FIGURE 11.

The above observation also reveals the fact that hom-sets are generated by basic arcs  $\epsilon : X \otimes X \rightarrow I$  and  $\delta : X \otimes X \rightarrow I$  together with their tensor product ampliations  $1_{X^m} \otimes \epsilon \otimes 1_{X^n}$  and  $1_{X^m} \otimes \delta \otimes 1_{X^n}$ .

In particular, a monoidal functor  $F$  on the tensor category  $\mathcal{K}_d$  is uniquely determined by morphisms  $F(\epsilon)$  and  $F(\delta)$ . When  $F : \mathcal{K}_d \rightarrow \mathcal{K}_{d'}$ , we should have  $F(\epsilon) = \lambda \epsilon$  and  $F(\delta) = \mu \delta$  with  $\lambda, \mu \in \mathbb{C}^\times$ . Since  $F$  must preserve hook identities, we are forced to set  $\lambda\mu = 1$ , which in

turn ensures the whole multiplicativity of  $F$ . In particular, we should have

$$d'1_I = F(\epsilon)F(\delta) = F(\epsilon\delta) = F(d1_I) = d1_I.$$

Thus Temperle-Lieb categories  $\mathcal{K}_d$  for different  $d$  are not equivalent as tensor categories.

The commutation relations of Temperley-Lieb algebra originally emerged in a model of statistical physics ([22]), which were later rediscovered by V. Jones as commutation relations among idempotents ([11]).

If we set  $e_i = h_i/d$ , these are idempotents and the Temperley-Lieb relations are equivalently described by the relations

$$\begin{cases} e_i e_j e_i = d^{-2} e_i & \text{if } |i - j| = 1, \\ e_i e_j = e_j e_i & \text{if } |i - j| \geq 2. \end{cases}$$

We remark here that this simple observation shows that  $\mathbb{C}[K_n, d_1] \cong \mathbb{C}[K_n, d_2]$  if  $d_1^2 = d_2^2$ .

Given an integer  $n \geq 1$ , let  $A_n$  be the algebra universally generated by  $\{h_1, \dots, h_{n-1}\}$  and the unit 1 with the Temperley-Lieb relations, which is referred to as the **Temperley-Lieb algebra**. (Rigorously speaking, we should use other symbols, say  $h'_i$ , instead of  $h_i$ .) By universality, we have the natural epimorphism  $A_n \rightarrow \mathbb{C}[K_n]$ , which turns out to be an isomorphism. (We reproduce somewhat simplified proofs in Appendix A.)

**Proposition 4.2** ([14, Theorem 4.3]). *The algebra  $\mathbb{C}[K_n, d]$  is universally generated by  $\{h_1, \dots, h_{n-1}\}$  and the unit 1 with the relations*

$$h_i^2 = dh_i, \quad h_i h_j = h_j h_i \quad (|i - j| \geq 2), \quad h_i h_{i \pm 1} h_i = h_i$$

*for  $1 \leq i, j \leq n - 1$ , whence it is identified with the Temperley-Lieb algebra  $A_n$ .*

*In particular, for  $n \geq 1$ , the obvious homomorphism  $A_n \rightarrow A_{n+1}$  of Temperley-Lieb algebras is injective.*

## 5. SEMISIMPLICITY AND FUSION RULES

Here we shall analyse the semisimplicity of the Temperley-Lieb categories together with their fusion rules, by looking into the structure of the algebra  $A_n = \text{End}(X^n)$  for  $n \geq 1$ .

Since  $A_2 = \langle 1, h_1 \rangle$  and  $h_1^2 = dh_1$ , the algebra  $A_2$  is semisimple if and only if  $d \neq 0$  and, if this is the case,  $A_2 = \mathbb{C}e_1 \oplus \mathbb{C}(1 - e_1) \cong \mathbb{C} \oplus \mathbb{C}$  with  $e_1 = h_1/d$  an idempotent. In other words, the linear subcategory generated by  $\{I, X, X^2\}$  is semisimple, with simple objects given by  $I$ ,  $X$  and  $X_2 \equiv f_2 X^2$ , where  $f_2 = 1 - e_1$  and  $e_1 X^2 \cong I$ ;  $X^2 \cong I \oplus X_2$ .

Note also that  $X_2$  is self-dual because  ${}^t f_2 = f_2$ . For the notational consistency, we also write  $f_1 = 1_X$  and  $X_1 = f_1 X = X$ .

Next, under the assumption of semisimplicity at the first stage, i.e.,  $d \neq 0$ , we see  $X^3 \cong (I \oplus X_2) \otimes X \cong X \oplus X_2 \otimes X$  in the tensor category  $\tilde{\mathcal{K}}_d$ . So we need to investigate how  $X_2 \otimes X$  is interrelated to  $X$ . By Frobenius reciprocity, we have

$$\begin{aligned} \text{Hom}(X, X_2 \otimes X) &\cong \text{Hom}(X \otimes X, X_2) \cong \text{Hom}(I \oplus X_2, X_2) = \text{End}(X_2), \\ \text{Hom}(X_2 \otimes X, X) &\cong \text{Hom}(X_2, X \otimes X) \cong \text{Hom}(X_2, I \oplus X_2) = \text{End}(X_2). \end{aligned}$$

Let  $\varphi_2 : X \rightarrow X_2 \otimes X$  and  $\psi_2 : X_2 \otimes X \rightarrow X$  be Frobenius transforms of  $1_{X_2} = f_2$  through the above isomorphisms (see Figure 14). Then any morphism  $X \rightarrow X_2 \otimes X$  is a scalar multiple of  $\varphi_2$  and similarly for  $\psi_2$ .

By a diagrammatic computation (Figure 12), we see that  $\psi_2 \varphi_2 = \lambda_2 1_X$  with  $\lambda_2 = d - d^{-1}$ . Thus, if  $\lambda_2 \neq 0$ ,  $X$  is a direct summand in  $X_2 \otimes X$  with the projection to  $X$  given by  $\lambda_2^{-1} \varphi_2 \psi_2 \in \text{End}(X_2 \otimes X)$  and the complementary subobject  $X_3$  of  $X_2 \otimes X$  given by the idempotent  $f_3 = f_2 \otimes 1_X - \lambda_2^{-1} \varphi_2 \psi_2$ . Since  $f_3$  is an idempotent in  $\text{End}(X_2 \otimes X) = (f_2 \otimes 1_X) \text{End}(X^3) (f_2 \otimes 1_X)$ , it is also an idempotent in  $A_3 = \text{End}(X^3)$  (satisfying  $f_3(f_2 \otimes 1_X) = (f_2 \otimes 1_X)f_3 = f_3$ ).

Since both of  $\text{Hom}(X, X_2 \otimes X) \cong \text{Hom}(X, X \oplus X_3)$  and  $\text{Hom}(X_2 \otimes X, X) \cong \text{Hom}(X \oplus X_3, X)$  are one-dimensional, we see that

$$\text{Hom}(X, X_3) = \text{Hom}(X_3, X) = \{0\},$$

while the parity condition implies the triviality of  $\text{Hom}(I, X_3)$ ,  $\text{Hom}(X_3, I)$ ,  $\text{Hom}(X_2, X_3)$  and  $\text{Hom}(X_3, X_2)$ . Note here that  $X^3 \cong 2X \oplus X_3$ .

If  $f_3 = 0$ , i.e.,  $X_3 = 0$ , then  $\text{End}(X^3) \cong M_2(\mathbb{C})$  is four-dimensional, a contradiction with  $\dim \text{End}(X^3) = 5$ . Thus  $f_3 \neq 0$  and the dimension estimate

$$2^2 + 1 \leq \dim \text{End}(X \oplus X) + \dim \text{End}(X_3) = \dim A_3 = 5$$

shows  $\text{End}(X_3) = \mathbb{C}f_3$ .

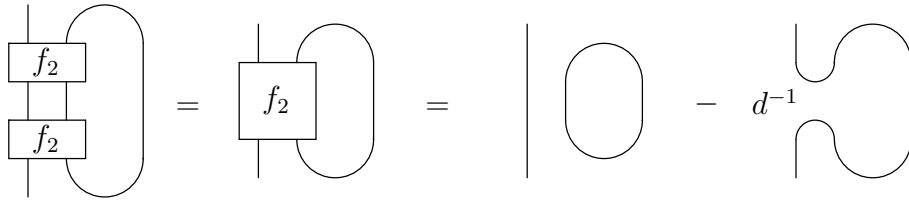


FIGURE 12.

Summarizing the discussion so far, under the assumption  $d \neq 0$  and  $d - d^{-1} \neq 0$ , the linear subcategory of  $\mathcal{K}_d$  generated by  $\{I, X, X^2, X^3\}$

is semisimple with the (isomorphism classes of) simple objects given by  $I$ ,  $X$ ,  $X_2$  and  $X_3$ .

The reasoning is applicable repeatedly and we arrive at the following induction scheme: Assume that idempotents  $f_k \in A_k$  are inductively defined up to  $k = n$  so that

- (i) the sequence  $\{f_k\}$  satisfies the recursive formula

$$f_{k+1} = f_k \otimes 1_X - \frac{\text{tr}(f_{k-1})}{\text{tr}(f_k)}(f_k \otimes 1_X)h_k(f_k \otimes 1_X)$$

with  $\text{tr}(f_k) \neq 0$  for  $1 \leq k < n$

- (ii) the linear subcategory generated by  $\{I, X, X^2, \dots, X^n\}$  is semisimple with inequivalent simple objects represented by  $X_k \equiv f_k X^{\otimes k}$  for  $0 \leq k \leq n$ ,

- (iii)  $X_k \otimes X \cong X_{k-1} \oplus X_{k+1}$  for  $1 \leq k < n$  with  $X_0 = I$ .

At this stage, we derive two consequences from the above hypotheses: Applying the Markov trace to the recursive formula for  $f_j$ , we have

$$\text{tr}(f_{k+1}) = d \text{tr}(f_k) - \text{tr}(f_{k-1})$$

for  $1 \leq k < n$  with  $\text{tr}(f_0) = 1$  and  $\text{tr}(f_1) = d$ . Consequently, by the choice  $d = q + q^{-1}$  with  $0 \neq q \in \mathbb{C}$ , we have

$$\text{tr}(f_k) = [k+1]_q = \frac{q^{k+1} - q^{-k-1}}{q - q^{-1}}.$$

From the fusion rule for  $X_k \otimes X$ , together with lattice path countings by the reflection technique, we obtain the multiplicity formula (see Figure 13)

$$X^{\otimes k} = \bigoplus_{j=0}^{[k/2]} \begin{bmatrix} k \\ j \end{bmatrix} X_{k-j}$$

for  $1 \leq k \leq n$ , where

$$\begin{bmatrix} k \\ j \end{bmatrix} = \binom{k}{j} - \binom{k}{j-1}.$$

( $\binom{k}{-1} = 0$  by definition.)

Here are explicit formula in lower cases:

$$\begin{aligned} X^2 &= X_2 \oplus I, \\ X^3 &= X_3 \oplus 2X, \\ X^4 &= X_4 \oplus 3X_2 \oplus 2I, \\ X^5 &= X_5 \oplus 4X_3 \oplus 5X. \end{aligned}$$

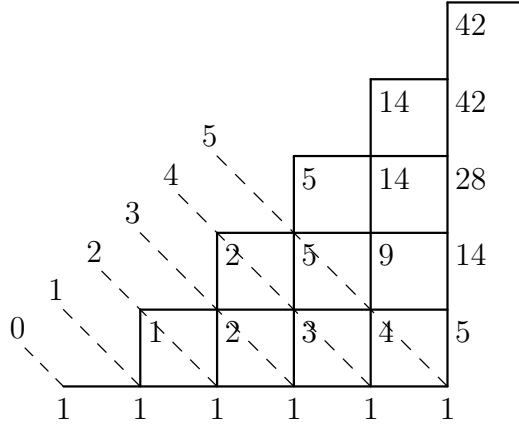


FIGURE 13.

We can now raise the induction stage one step further: By Frobenius reciprocity and the induction hypothesis on the fusion rule, we have

$$\text{Hom}(X_k, X_n \otimes X) \cong \text{Hom}(X_{k-1} \oplus X_{k+1}, X_n) = \{0\},$$

$$\text{Hom}(X_n \otimes X, X_k) \cong \text{Hom}(X_n, X_{k-1} \oplus X_{k+1}) = \{0\}$$

for  $1 \leq k \leq n-2$ , which means that no  $X_k$ -component appears in  $X_n \otimes X$  for  $1 \leq k \leq n-2$ . On the other hand, again by Frobenius reciprocity and the fusion rule assumption,

$$\text{Hom}(X_{n-1}, X_n \otimes X) \cong \text{Hom}(X_{n-1} \otimes X, X_n) \cong \text{Hom}(X_{n-2} \oplus X_n, X_n) = \text{End}(X_n),$$

$$\text{Hom}(X_n \otimes X, X_{n-1}) \cong \text{Hom}(X_n, X_{n-1} \otimes X) \cong \text{Hom}(X_n, X_{n-2} \oplus X_n) = \text{End}(X_n).$$

Thus, as Frobenius transforms of  $1_{X_n} = f_n$ , we can define non-zero morphisms  $\varphi_n : X_{n-1} \rightarrow X_n \otimes X$  and  $\psi_n : X_n \otimes X \rightarrow X_{n-1}$  (Figure 14).

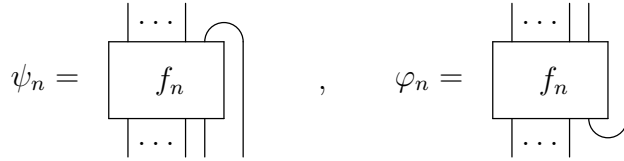


FIGURE 14.

Then, by manipulating diagrams (Figure 15, Figure 16), we see  $\psi_n \varphi_n = \lambda_n 1_{X_n}$  with  $\lambda_n = \text{tr}(f_n)/\text{tr}(f_{n-1})$ .

Therefore, if  $\text{tr}(f_n) \neq 0$ , then we can define an idempotent  $f_{n+1} \in A_{n+1}$  by the formula

$$f_{n+1} = f_n \otimes 1_X - \frac{\text{tr}(f_{n-1})}{\text{tr}(f_n)} \varphi_n \psi_n = f_n \otimes 1_X - \frac{\text{tr}(f_{n-1})}{\text{tr}(f_n)} (f_n \otimes 1_X) h_n(f_n \otimes 1_X)$$

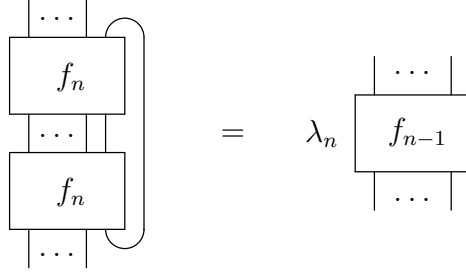


FIGURE 15.

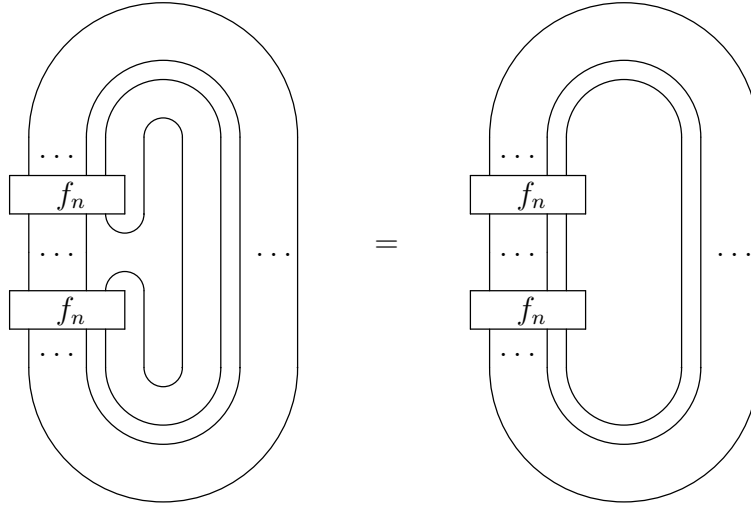


FIGURE 16.

(Figure 17) with the associated subobject  $X_{n+1} = f_{n+1}X^{\otimes(n+1)}$  and we reach the direct sum decomposition  $X_n \otimes X \cong X_{n-1} \oplus X_{n+1}$ . Since both of  $\text{Hom}(X_{n-1}, X_n \otimes X)$  and  $\text{Hom}(X_n \otimes X, X_{n-1})$  are one-dimensional, we have

$$\text{Hom}(X_{n-1}, X_{n+1}) = \{0\} = \text{Hom}(X_{n+1}, X_{n-1}),$$

whereas the triviality of  $\text{Hom}(X_n, X_{n+1})$  and  $\text{Hom}(X_{n+1}, X_n \otimes X)$  is a consequence of parity discrepancy.

From the multiplicity formula for  $X^{\otimes n}$  and the fusion rule  $X_n \otimes X \cong X_{n-1} \oplus X_{n+1}$  with  $\text{Hom}(X_k, X_{n+1}) = \{0\} = \text{Hom}(X_{n+1}, X_k)$  for  $1 \leq k \leq n$ , we obtain the decomposition

$$X^{\otimes(n+1)} = \bigoplus_{j=0}^{[(n+1)/2]} \begin{bmatrix} n+1 \\ j \end{bmatrix} X_{n+1-j}.$$

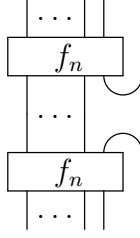


FIGURE 17.

At this point, we have no information on the simplicity of the new stuff  $X_{n+1}$  yet. The above decomposition, however, gives rise to the following dimension identity

$$\dim \operatorname{End}(X^{\otimes(n+1)}) = \dim \operatorname{End}(X_{n+1}) - 1 + \sum_{j=0}^{[(n+1)/2]} \begin{bmatrix} n+1 \\ j \end{bmatrix}^2,$$

which particularly implies  $f_{n+1} \neq 0$ .

Now we conclude  $\operatorname{End}(X_{n+1}) = \mathbb{C}f_{n+1}$  from the combinatorial formula below, which is obtained by folding halfway in the following well-known binomial identity (see [17, Chapter 5] for example)

$$\sum_k \binom{m}{a+k} \binom{n}{b+k} = \binom{m+n}{m-a+b} = \binom{m+n}{n+a-b}.$$

**Lemma 5.1** ([11, 9]). *For a positive integer  $n$ , we have*

$$\sum_{j=0}^{[n/2]} \begin{bmatrix} n \\ j \end{bmatrix}^2 = \frac{1}{n+1} \binom{2n}{n}.$$

For  $n = 5$ , this means

$$1^2 + 4^2 + 5^2 = 42.$$

As a conclusion of induction arguments so far, we have

**Proposition 5.2.** *Express  $d$  in the form  $d = q + q^{-1}$  with  $q \in \mathbb{C}^\times$ . Assume that  $[k]_q \neq 0$  for  $1 \leq k \leq n$ . Then the linear subcategory of  $\mathcal{K}_d$  generated by  $\{I, X, \dots, X^{\otimes n}\}$  is semisimple and a representative simple objects  $X_k$  ( $k = 1, 2, \dots, n$ ) in  $\tilde{\mathcal{K}}_d$  are inductively defined so that  $X_k \otimes X \cong X_{k-1} \oplus X_{k+1}$  ( $1 \leq k < n$ ).*

*The subobject  $X_k$  appears only once in  $X^{\otimes k}$  and the associated idempotent  $f_k \in \operatorname{End}(X^{\otimes k})$  is inductively defined by the Wenzl's formula*

$$f_{k+1} = f_k \otimes 1_X - \frac{[k]_q}{[k+1]_q} (f_k \otimes 1_X) h_k (f_k \otimes 1_X)$$



with the trace value given by  $\text{tr}(f_k) = [k + 1]_q$ .

**Corollary 5.3.** *If  $q^2$  is not a proper root of unity, the Temperley-Lieb category  $\mathcal{K}_d$  is essentially semisimple with its fusion rule given by the Clebsh-Gordan rule*

$$X_j \otimes X_k \cong X_{|j-k|} \oplus X_{|j-k|+2} \oplus \cdots \oplus X_{j+k}.$$

If  $q^2$  is an  $l$ -th primitive root of unity, the non-degeneracy of the Markov trace ceases at the algebra  $\text{End}(X^{\otimes(l-1)})$ . It is then customary to take a quotient of the tensor category  $\mathcal{K}_d$ : Let

$$\text{Ker}(X^{\otimes m}, X^{\otimes n}) = \{f \in \text{Hom}(X^{\otimes m}, X^{\otimes n}); \langle fg \rangle = 0 \text{ for } g \in \text{Hom}(X^{\otimes n}, X^{\otimes m})\}.$$

Then these constitute an ideal of  $\mathcal{K}_d$  in the sense that

- (i)  $\text{Hom}(X^{\otimes m}, X^{\otimes n}) \text{Ker}(X^{\otimes l}, X^{\otimes m}) \text{Hom}(X^{\otimes k}, X^{\otimes l}) \subset \text{Ker}(X^{\otimes k}, X^{\otimes n})$   
and
- (ii)  $\text{Hom}(X^{\otimes m'}, X^{\otimes n'}) \otimes \text{Ker}(X^{\otimes m}, X^{\otimes n}) \otimes \text{Hom}(X^{\otimes m''}, X^{\otimes n'') \subset \text{Ker}(X^{\otimes(m'+m+m'')}, X^{\otimes(n'+n+n'')})$ ,

where the former is a consequence of the trace property of  $\text{tr}(\cdot)$  and the latter is checked by means of the conditional expectation  $\text{End}(X^{m+n}) \rightarrow \text{End}(X^m)$  with respect to the inclusion  $\text{End}(X^m) \cong \text{End}(X^m) \otimes 1 \subset \text{End}(X^{m+n})$ .

The quotient tensor category  $\bar{\mathcal{K}}_d$  is then defined so that objects are the same with those for  $\mathcal{K}_d$  (but we shall use the bar notation to indicate objects in  $\bar{\mathcal{K}}_d$ ) and hom-sets are given by  $\text{Hom}(\bar{Y}, \bar{Z}) = \text{Hom}(Y, Z) / \text{Ker}(Y, Z)$  with the monoidal structure on  $\bar{\mathcal{K}}_d$  inherited from  $\mathcal{K}_d$ . The resultant tensor category is then referred to as the **reduced Temperley-Lieb category**.

Now the following is immediate from our discussions so far.

**Proposition 5.4.** *Assume that  $q^2$  is an  $l$ -th primitive root of unity. Then*

$$\text{Ker}(X^{\otimes k}) = \begin{cases} \{0\} & \text{if } k < l-1, \\ \mathbb{C}f_{l-1} & \text{if } k = l-1 \end{cases}$$

and  $\text{Ker}(X^{\otimes m}, X^{\otimes n})$  is monoidally generated by  $f_{l-1}$ .

The reduced Temperley-Lieb category is semisimple with simple objects given by  $\{\bar{X}_k\}_{0 \leq k \leq l-2}$  with the recursive formula

$$\bar{X}_k \otimes \bar{X} = \begin{cases} \bar{X}_{k-1} \oplus \bar{X}_{k+1} & \text{if } k < l-2, \\ \bar{X}_{l-3} & \text{if } k = l-2. \end{cases}$$

**Corollary 5.5.** *The fusion rule for  $\{\bar{X}_k\}$  is given by the truncated Clebsh-Gordan rule:*

$$\bar{X}_j \otimes \bar{X}_k \cong \bar{X}_{|j-k|} \oplus \bar{X}_{|j-k|+2} \oplus \cdots \oplus \bar{X}_m,$$

where

$$m = \begin{cases} j + k & \text{if } j + k \leq l - 2, \\ 2(l - 2) - (j + k) & \text{if } j + k \geq l - 2. \end{cases}$$

*Remark* . The kernel of  $\mathcal{K}_d$  is characterized in [10] as the unique monoidal ideal.

## 6. POSITIVITY IN TEMPERLEY-LIEB CATEGORIES

We shall now clarify the condition when the Temperley-Lieb category is a  $C^*$ -tensor category. Recall that a linear category  $\mathcal{L}$  is a  **$C^*$ -category** if hom-sets are Banach spaces and we are given conjugate linear maps (denoted by  $*$  and referred to as a star operation) on hom-sets  $\text{Hom}(Y, Z) \ni f \mapsto f^* \in \text{Hom}(Z, Y)$  satisfying (i)  $(f^*)^* = f$ , (ii)  $(fg)^* = g^*f^*$ , and (iii)  $\|f^*f\| = \|f\|^2$  for  $g : X \rightarrow Y$ ,  $f : Y \rightarrow Z$ . When hom-sets are finite-dimensional, a more algebraic formulation is possible (see [9, Appendix]): given a star operation satisfying (i), (ii) and the condition that  $f^*f = 0$  implies  $f = 0$ , there is the unique  $C^*$ -norm fulfilling (iii).

A  **$C^*$ -tensor category** (or tensor  $C^*$ -category) is a (strict) tensor category which is a  $C^*$ -category at the same time with the common underlying linear structure such that two structures are compatible in the sense that  $(f \otimes g)^* = f^* \otimes g^*$  for morphisms  $f, g$ . (when the associativity transformations are explicit, they are assumed to be unitary with respect to the star operation).

A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between two (strict) tensor categories is said to be **monoidal** if  $F(I_{\mathcal{C}}) = I_{\mathcal{D}}$ ,  $F(X \otimes Y) = F(X) \otimes F(Y)$  for objects  $X, Y$  and  $F(f \otimes g) = F(f) \otimes F(g)$  for morphisms  $f, g$ .

Two tensor categories are said to be **monoidally equivalent** if we can find a monoidal functor between these tensor categories which gives an equivalence of categories; if the functor  $F$  is fully faithful in the sense that  $F$  gives isomorphisms on hom-sets and any object of  $\mathcal{D}$  is isomorphic to  $F(X)$  for some object  $X$  in  $\mathcal{C}$ .

When  $\mathcal{C}$  and  $\mathcal{D}$  are  $C^*$ -tensor categories, a monoidal functor is said to be  **$C^*$ -monoidal** if  $F(f)^* = F(f^*)$  for any morphism  $f$  in  $\mathcal{C}$ .

*Remark* . Our definition of monoidality is the one usually referred to as being strict. Since any non-strict monoidal functor is changed to be strict by replacing tensor categories with equivalent ones, there are no essential differences.

The following reveals the universality of Temperley-Lieb categories concerning self-dual objects in tensor categories.

**Lemma 6.1.** *Let  $Y$  be a self-dual object in a tensor category  $\mathcal{T}$  with the associated morphisms  $\epsilon_Y : Y \otimes Y \rightarrow I$ ,  $\delta_Y : I \rightarrow Y \otimes Y$  and suppose that  $\epsilon_Y \delta_Y = d1_I$  with  $d \in \mathbb{C}^\times$ .*

*Then the correspondence  $\epsilon \mapsto \epsilon_Y$ ,  $\delta \mapsto \delta_Y$  is extended to a monoidal functor from the Temperley-Lieb category  $\mathcal{K}_d$  to  $\mathcal{T}$ .*

*Proof.* By a single arc, we shall mean a morphism of the form  $1 \otimes \epsilon \otimes 1$  or  $1 \otimes \delta \otimes 1$ . Given an arc  $a$  in  $\mathcal{K}_d$ , we denote by  $F(a)$  the morphism in  $\mathcal{T}$  defined by

$$F(a) = \begin{cases} 1_{Y^m} \otimes \epsilon_Y \otimes 1_{Y^n} & \text{if } a = 1_{X^m} \otimes \epsilon \otimes 1_{X^n}, \\ 1_{Y^m} \otimes \delta_Y \otimes 1_{Y^n} & \text{if } a = 1_{X^m} \otimes \delta \otimes 1_{X^n}. \end{cases}$$

As in the proof of the generating property of elementary diagrams, we see that each diagram  $D \in K_{m,n}$  can be expressed as a (loopless) composition  $a_1 a_2 \dots a_M$  of single arcs  $\{a, a_2, \dots, a_M\}$ . Furthermore, given another loopless presentation  $D = b_1 \dots b_N$  by arcs, these are related by repeating one of the fundamental planar identities (Figure 18) locally.

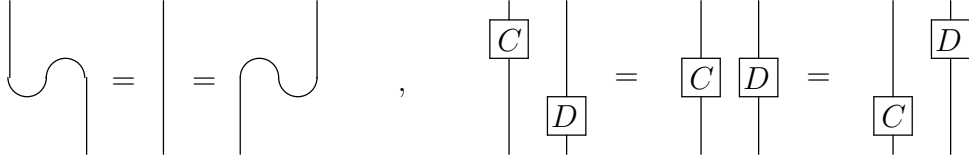


FIGURE 18.

In the process of applying these identities, the composed morphism  $F(a_1) \dots F(a_M)$  remains unchanged because of the validity of the corresponding rigidity identities in  $\mathcal{T}$ .

Thus, for a diagram  $D \in K_{m,n}$ , the morphism  $F(D) \in \text{Hom}(Y^{\otimes m}, Y^{\otimes n})$  is well-defined by the formula

$$F(D) = F(a_1)F(a_2) \dots F(a_M),$$

where  $D = a_1 a_2 \dots a_M$  is a loopless presentation of  $D$  as a product of single arcs  $a_1, a_2, \dots, a_M$ . By linearity,  $F$  is extended to linear maps

$$\text{Hom}(X^{\otimes m}, X^{\otimes n}) \rightarrow \text{Hom}(Y^{\otimes m}, Y^{\otimes n}).$$

By the choice of  $d$ , these linear maps preserve multiplications and therefore it gives rise to a functor  $\mathcal{K}_d \rightarrow \mathcal{T}$ , which, by the construction, is monoidal and satisfies  $F(\epsilon) = \epsilon_Y$ ,  $F(\delta) = \delta_Y$ .  $\square$

**Proposition 6.2.** *The Temperley-Lieb category  $\mathcal{K}_d$  is a  $C^*$ -tensor category if and only if  $d = \pm(q + q^{-1})$  with  $q > 0$  or  $q = e^{i\pi/n}$  ( $n \geq 3$ ).*

For such a value of  $d$ , the  $C^*$ -tensor category structure on  $\mathcal{K}_d$  is unique up to  $C^*$ -monoidal equivalences.

An explicit choice is given by

$$D^* = \left( \frac{d}{|d|} \right)^{\sharp(D)} D',$$

where  $D'$  denotes the diagram obtained from  $D$  up-side down (i.e., the reflection of  $D$  with respect to a horizontal line) and  $\sharp(D)$ , called the arc index of  $D$ , is the difference of the number of  $\epsilon$ 's and  $\delta$ 's inside  $D$ .

*Proof.* Assume that  $\mathcal{K}_d$  is a  $C^*$ -tensor category. Since the star operation preserves the central decomposition in a  $C^*$ -algebra, the decomposition  $\text{End}(X \otimes X) = \mathbb{C}(1-e) + e$  with  $e = d^{-1}h$  implies  $e^* = e$ . Consequently, we have  $e_j^* = e_j$  for any  $j \geq 1$  and the relation  $d^2 e_1 e_2 e_1 = e_1$  compels the positivity of  $d^2$ , i.e.,  $d \in \mathbb{R}^\times$ .

The inductive analysis on semisimplicity now shows that the Jones-Wenzl idempotents  $f_k$  are (orthogonal) projections and  $[2][n]/[n+1] \geq 0$  if  $[k] \neq 0$  for  $k \leq n+1$  because  $[2][n]/[n+1](f_n \otimes 1_X)e_n(f_n \otimes 1_X)$  is a subprojection of  $f_{n+1}$ . The condition is then equivalent to (i)  $(d/|d|)^{k+1}[k] > 0$  for all  $k$  or (ii)  $(d/|d|)^{k+1}[k] > 0$  for  $1 \leq k < l$  with  $[l] = 0$  for a positive integer  $l \geq 3$ . It is then immediate to see that these conditions are equivalent to the ones given in the statement of the proposition.

Since both of  $\text{Hom}(X \otimes X, I)$  and  $\text{Hom}(I, X \otimes X)$  are one-dimensional, we should have  $\epsilon^* = c\delta$  with  $c \in \mathbb{C}^\times$  and the positivity of  $\epsilon^*\epsilon = cde$  shows that  $c$  is a real number, which is positive or negative according to the signature of  $d$ . In particular,  $\delta^* = c^{-1}\epsilon$ .

Conversely, assume that  $d \in \mathbb{R}$  is in the range specified above and let  $r$  be a positive real.

For each pair  $(m, n)$ , define the conjugate-linear map  $\text{Hom}(X^m, X^n) \rightarrow \text{Hom}(X^n, X^m)$  so that

$$D^* = \left( \frac{rd}{|d|} \right)^{\sharp(D)} D'$$

for  $D \in K_{m,n}$ . (We have particularly  $\epsilon^* = rd|d|^{-1}\delta$  and  $h_j^* = h_j$ .) From the definition, we have  $(f \otimes g)^* = f^* \otimes g^*$  for morphisms  $f, g$  in  $\mathcal{K}_d$ . The map is involutive because of  $\sharp(D') = -\sharp(D)$  and it satisfies  $(fg)^* = g^*f^*$ , which follows from the additivity of arc index:  $\sharp(CD) = \sharp(C) + \sharp(D)$  for a composable pair  $(C, D)$  of diagrams.

In this way, we have defined a  $*$ -operation in the tensor category  $\mathcal{K}_d$ . Note here that only the reality of  $r$  and  $d$  is used up to now. Notice also that the Jones-Wenzl idempotents  $f_n$  are hermitian, i.e.,  $f_n^* = f_n$ ,

which is checked by the recursive formula for them by using the reality of  $d$  and hence of  $[n]$ .

We shall now check the positivity of the  $*$ -operation in question, which will be achieved by seeing that there are plenty of positive (unitary) representations of  $\text{End}(X^n)$ . By our choice of signature in the definition of star operation, we see that the morphism

$$r^{1/2} \left| \frac{\text{tr}(f_{n-1})}{\text{tr}(f_n)} \right|^{1/2} \varphi_n : X_{n-1} \rightarrow X_n \otimes X$$

gives a realization of  $X_{n-1}$  as an orthogonal component of  $X_n \otimes X$ :

$$r \left| \frac{\text{tr}(f_{n-1})}{\text{tr}(f_n)} \right| \varphi_n^* \varphi_n = \frac{\text{tr}(f_{n-1})}{\text{tr}(f_n)} \psi_n \varphi_n = f_{n-1},$$

where, in the first equality, we have calculated as

$$\begin{aligned} \varphi_n^* &= ((f_n \otimes 1_X)(1_{X^{n-1}} \otimes \delta))^* = (1_{X^{n-1}} \otimes \delta^*)(f_n \otimes 1_X) \\ &= \frac{|d|}{rd} (1_{X^{n-1}} \otimes \epsilon)(f_n \otimes 1_X) = \frac{|d|}{rd} \psi_n. \end{aligned}$$

Notice also that, by the assumption on  $d$ , the signature of  $\text{tr}(f_k) = [k+1]$  alternates as  $k$  increases until it vanishes.

Thus, the path basis in the representation space  $\text{Hom}(X_k, X^{\otimes n})$  of  $\text{End}(X^{\otimes n})$  ( $0 \leq k \leq n$  with  $k \equiv n \pmod{2}$ ) is orthonormal with respect to the inner product  $(\cdot|\cdot)$  defined by  $f^*g = (f|g)f_k$ .

By the obvious decomposition

$$\text{Hom}(X^m, X^n) \cong \bigoplus_k \text{Hom}(X_k, X^n) \otimes \text{Hom}(X^m, X_k),$$

the previous observation on representations of  $\text{End}(X^{\otimes n})$  reveals that the category  $\mathcal{K}_d$  turns out to be a  $C^*$ -tensor category.

Finally, we show the uniqueness of  $C^*$ -structures. Let  $*$  denote a star operation in  $\mathcal{K}_d$  which makes  $\mathcal{K}_d$  into a  $C^*$ -tensor category. Given  $\lambda \in \mathbb{C}^\times$ , define a monoidal functor  $F : \mathcal{K}_d \rightarrow \mathcal{K}_d$  so that

$$F(\epsilon) = \lambda\epsilon, \quad F(\delta) = \lambda^{-1}\delta,$$

i.e.,

$$F(D) = \lambda^{\sharp(D)} D$$

for a diagram  $D$ . Then  $F$  is an automorphism of the tensor category  $\mathcal{K}_d$  and the new star operation  $\star$  in  $\mathcal{K}_d$ , which again gives a compatible  $C^*$ -structure in  $\mathcal{K}_d$ , is defined by the relation

$$F(f)^\star = F(f^*) \quad \text{for a morphism } f \text{ in } \mathcal{K}_d.$$

Then, on the level of diagrams, we have the explicit formula

$$D^* = |\lambda|^{-2\sharp(D)} D^*.$$

Thus, compatible structures of  $C^*$ -tensor category on  $\mathcal{K}_d$  are unique up to  $C^*$ -monoidal equivalences.  $\square$

*Remark .* The duality isomorphism  $X \cong X^{**}$  is given by the identity morphism  $1_X$  for the case  $d > 0$ , whereas it is  $-1_X$  if  $d < 0$ , resulting in the positive value  $|d|$  as the quantum dimension of  $X$  in either cases. A bit more analysis (see [8], [16] for example) shows that  $\mathcal{K}_d$  and  $\mathcal{K}_{-d}$  are related to each other by twisting associativity isomorphisms in monoidal structure with respect to a non-trivial 3-cocycle of the group  $\mathbb{Z}^2$ . On the level of operator algebras, this is interpreted as twisting a generating bimodule with respect to an outer automorphism  $\alpha$  of a factor  $N$  satisfying (i)  $\alpha \circ \alpha = \text{Ad } u$  (ii) and  $\alpha(u) = -u$  with  $u$  a unitary in  $N$ .

## APPENDIX A. UNIVERSALITY PROPERTY

We shall here present a proof of the fact that the Kauffman's planar algebra  $\mathbb{C}[K_n, d]$  is identified with the Temperley-Lieb algebra  $A_n$ , which is universally generated by elements  $\{h_1, \dots, h_{n-1}\}$  with the relations of Temperley-Lieb. Since there is a natural homomorphism  $\pi : A_n \rightarrow \mathbb{C}[K_n, d]$  by universality, the problem is in checking the bijectivity of  $\pi$ , which was claimed in [13] with more accounts supplied in [14, Theorem 4.3] (cf. [4] also). The proof obviously consists of two parts: the surjectivity of  $\pi$  and the injectivity of  $\pi$ . The former is the generating property of elementary diagrams in the algebra  $\mathbb{C}[K_n, d]$ , while the latter is reduced to the problem of counting reduced words of generators.

As for the generating property, the following would not be the shortest proof, compared with the one given in [23, Theorem XII.3.2] for instance, but has the advantage that it produces reduced words. (The Jones' normal form is then obtained shearing positions of mutually commutable elementary diagrams as indicated by [14, Figure 16].

For discussions of the proof, we introduce some terminologies first. Given a diagram  $D$  in  $K_n$ , a string inside  $D$  is called a through string if it connects upper and lower vertices, and an arc otherwise. A through string is said to be vertical if it connects vertices in the same horizontal position. A handle is, by definition, an arc which connects neighboring vertices.

We shall apply an induction of trying to increase the number of handles inside relevant diagrams to get the generating property. If a

diagram  $D$  contains a vertical through string, then  $D$  is of the form  $D' \otimes D''$  with  $D' \in K_{n'}$  and  $D'' \in K_{n''}$ , whence the problem is reduced to diagrams of less strings.

Consider the case that  $D$  contains a through string connecting two vertices  $i < j$ , say  $i$  on the top and  $j$  on the bottom. Then  $j - i$  is an even number (otherwise there would appear unconnected vertices) and we can apply waving to the string (i.e., the string is deformed so that it repeats local maxima and minima alternately) after separating it from the other strings by stretching out sufficiently. Then there arise three patterns depending on the position where the vertex  $i + 1$  on the top is terminated: If it ends at a bottom vertex numbered by  $k$ , then  $k > j$  and the waving of this second through string, together with the waving of the first string, gives us couplings of handles during the horizontal interval  $[i, j]$ . If we further stretch the waving of the second string on the unoverlapping interval  $[j + 1, k]$ , push down the strings tied to vertices in the interval  $[1, i - 1]$ , and pull up the strings ending at vertices in the interval  $[k + 1, n]$  sufficiently enough so that  $D$  becomes the composition of three diagrams in  $K_n$  as indicated by dotted horizontal lines in Figure 19. Then the middle diagram is apparently a product of (mutually commuting) elementary diagrams, whereas the remaining diagrams contain vertical through strings.

Otherwise, the string starting from  $i + 1$  forms an arc ending at the vertex  $k$  with  $k > i + 1$ . Though we need to further divide into two patterns depending on the relative position of  $j$  and  $k$ , a similar decomposition is possible as indicated by pictures (Figure 20, Figure 21) and we are again reduced to diagrams of less strings.

Finally, there remains the case that  $D$  contains no through string. If there is an arc which is not a handle, we see  $D$  containing a part of continuing handles surrounded by one arc. Then, by waving the surrounding arc, and a similar rearrangement as above allows us to identity  $D$  with the composition of two diagrams such that one of them is again a product of (commuting) elementary diagrams and the other has more handles than the original diagram (Figure 22). Repeating the same procedure, we end up with diagrams in which all the arcs are handles, thus again a product of commuting elementary diagrams.

To see the injectivity of the map  $\pi : A_n \rightarrow \mathbb{C}[K_n]$ , we use the dimension estimate of  $A_n$  here: According to V. Jones, by a **word** in  $A_n$ , we shall mean a product  $h_{i_1} h_{i_2} \dots h_{i_k}$  where  $h_{i_1}, \dots, h_{i_k} \in \{h_1, \dots, h_{n-1}\}$  with two words identified if we can relate them each other by applying the commutativity  $h_i h_j = h_j h_i$  ( $|i - j| \geq 2$ ) to their ingredients. Thus  $h_1 h_3 h_1 h_2 = h_3 h_1 h_1 h_2$  as a word for example. The length of a word is, by definition, the number of  $h_i$ 's appearing in the word. A word is

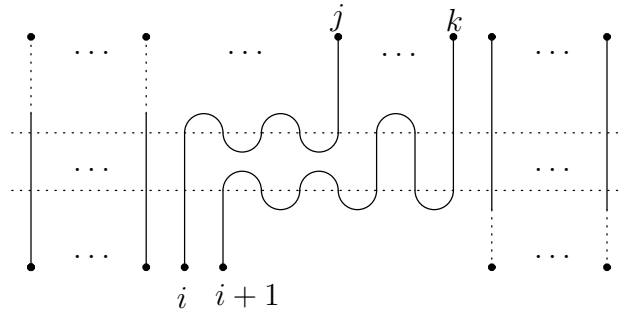


FIGURE 19.

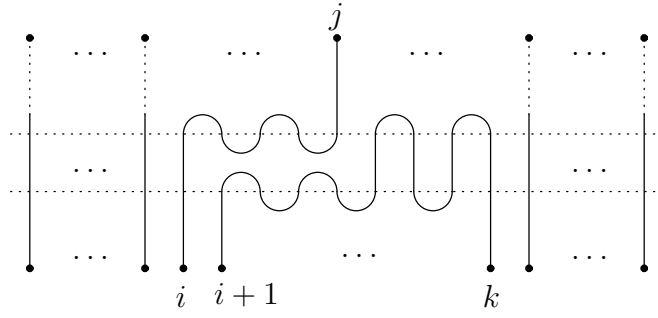


FIGURE 20.

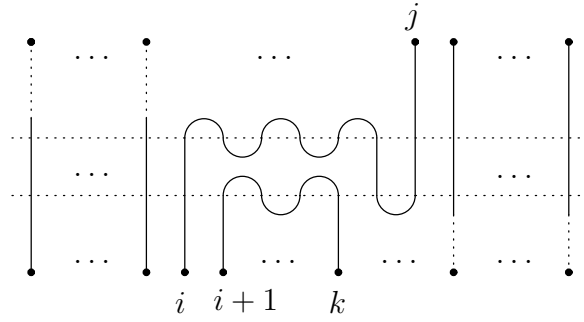


FIGURE 21.

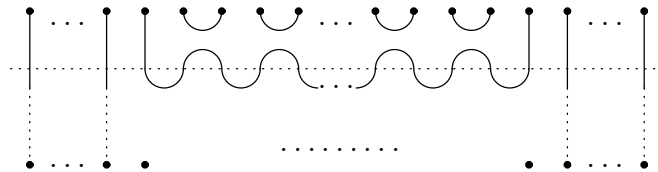


FIGURE 22.

said to be **reduced** if its length is minimal under the replacements of



$h_i h_{i\pm 1} h_i$  to  $h_i$  and  $h_i^2$  to  $h_i$ . Thus any word is equal to a reduced one up to multiplication of powers of  $d$ .

From the commutation relations,  $h_m$  with  $m$  the maximal index appears only once in a reduced word. According to V. Jones, a reduced word is further relocated so that  $h_m$  is placed at the rightest end. Then, after the point  $h_m$ , there follows a sequence of the form  $h_m h_{m-1} \dots h_l$  with  $l \leq m$ . On the left of the block of this sequence, we are left a reduced word consisting of elements in  $\{h_1, h_2, \dots, h_{m-1}\}$ , for which we can apply the same procedure to get eventually the form

$$(h_{i_1} h_{i_1-1} \dots h_{j_1})(h_{i_2} h_{i_2-1} \dots h_{j_2}) \dots (h_{i_k} h_{i_k-1} \dots h_{j_k}),$$

with  $i_1 \geq j_1, i_2 \geq j_2, \dots, i_k \geq j_k$  and  $1 \leq i_1 < i_2 < \dots < i_k \leq n-1$  (the case  $k=0$  corresponding to the empty word).

Since this is assumed to be a reduced word, we should have  $1 \leq j_1 < j_2 < \dots < j_k \leq n-1$  as well. In fact, if  $j_1 \geq j_2$  for example, we should have  $h_{j_1}$  appearing in the block  $h_{i_2} h_{i_2-1} \dots h_{j_2}$  by  $j_1 \leq i_1 < i_2$  and hence a reduction of the form  $h_{j_1} h_{j_1+1} h_{j_1} = h_{j_1}$  takes place, reducing the word length.

**Proposition A.1** ([11, §4]). *Any word in  $A_n$  is equal to a reduced one up to scalar multiplications and the number of reduced words in  $A_n$  is given by the Catalan number*

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

**Corollary A.2** (cf. [14, Theorem 4.3]). *The algebra  $\mathbb{C}[K_n, d]$  is universally generated by  $\{h_1, \dots, h_{n-1}\}$  and the unit 1 with the relations*

$$h_i^2 = dh_i, \quad h_i h_j = h_j h_i \quad (|i-j| \geq 2), \quad h_i h_{i\pm 1} h_i = h_i$$

for  $1 \leq i, j \leq n-1$ , whence it is identified with the Temperley-Lieb algebra  $A_n$ .

In particular, for  $n \geq 1$ , the obvious homomorphism  $A_n \rightarrow A_{n+1}$  of Temperley-Lieb algebras is injective.

In connection with the identification  $A_n = \mathbb{C}[K_n, d]$ , the following answers how Jones' reduced words are related to Kauffman's diagrams, which also constitutes a part of [14, Theorem 4.3]. We shall present a proof as a continuation of discussions so far.

**Proposition A.3.** *The set  $K_n$  of Kauffman diagrams is exactly the image of the set of reduced words in  $A_n$  under the natural isomorphism.*

*Proof.* We need to show that reduced words contain no loops when they are computed as compositions of diagrams in  $K_n$ . We shall check this by an induction on the number of blocks in the Jones normal form. Let

$h_m h_{m-1} \dots h_l$  be the last block in a Jones reduced word. As a diagram, this descending sequence is given by Figure 23. Since the previous blocks constitute a reduced word of Jones form with the number of blocks reduced, it contains no loops inside by the induction hypothesis and therefore the composition with the block  $h_m \dots h_l$  remains loopless.  $\square$

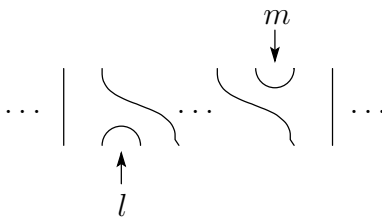


FIGURE 23.

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